Bounds on Maximum Weight Directed Cut

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Definitions

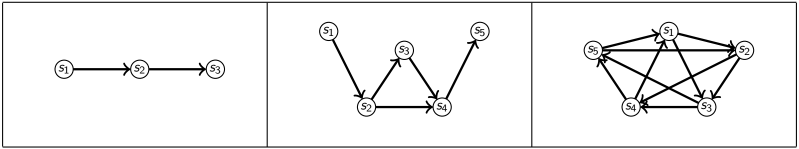
Definition

For every ordered bipartition (X, Y) of the vertex set of D, a directed cut (X, Y) is the bipartite subgraph induced by arcs go from X to Y. The weight of the directed cut (X, Y) is $w(X, Y) = \sum_{uv \in (X,Y)} w(uv)$.

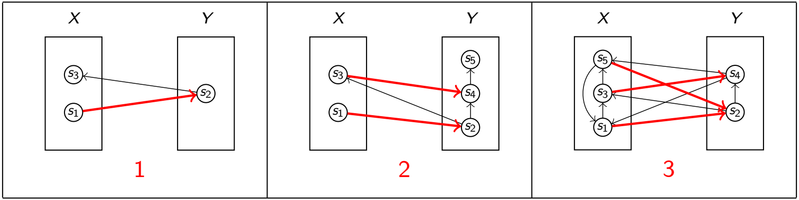
Problem (Max Weight Dicut Problem)

Given a weighted digraph $D = (V, E, w)$, find the maximum weight of a directed cut in D.

Examples



What is the directed max-cut for these digraphs? Why?



Weighted Eulerian Digraphs

Let $D = (V, E, w)$ be a weighted digraph. Let $w^+(v)$ denote the sum of the weight on the arcs leaving v in D (we call $w^+(v)$ the *out-weight* of v in D) and let $w^-(v)$ denote the sum of the weight on the arcs entering v in D (we call $w^-(v)$ the *in-weight* of v in D). We denote by $mac(D)$ the weight of a maximum directed cut in D.

Lemma

If $w^+(v) = w^-(v)$ for all $v \in V(D)$, then $mac(D) = \frac{mac(G)}{2}$ (where G is the underlying graph of D obtained by replacing every arc with an edge with the same weight).

Proof.

□

Regular tournament

A tournament is an orientation of a complete graph.

Theorem 1: If T is a regular tournament of order n then $mac(T) \approx \frac{m}{4}$.

Proof: As T is eulerian we note that

$$mac(T) = \frac{mac(K_n)}{2} \approx \frac{1}{2} \cdot \frac{m}{2} = \frac{m}{4}.$$

If $w^+(v) \neq w^-(v)$ for some v

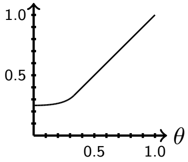
Let D be an arc-weighted digraph and let $w(D)$ denote the sum of all weights in D, i.e. $w(D) = \sum_{a \in A(D)} w(a)$.

Let $\theta(D) = \frac{\sum_{v \in V(D)} \max\{0, w^+(v) - w^-(v)\}}{w(D)}$. Then, $\theta(D) \in [0,1]$.

$\theta(D) = 0 \iff w^+(v) = w^-(v)$ for all $v \in V(D)$.
$\theta(D) = 1 \iff$ *D is a directed cut.*

Theorem 2, [1]: *mac*$(D) \geq l(\theta(D)) \cdot w(D)$, where

$$l(\theta) = \begin{cases} \left(\frac{1}{4} + \frac{\theta^2}{4(1-2\theta)}\right) & \text{if } 0 \leq \theta < 1/3; \\ \theta & \text{if } 1/3 \leq \theta \leq 1. \end{cases}$$



The bound is tight.

If $\theta(D) \geq 1/3$ then we simply put all vertices, x, with $w^+(x) > w^-(x)$ in X and all other vertices in Y.

Theorem 2, Sketch of proof

If $0 \leq \theta(D) < 1/3$, then the proof uses a probabilistic argument. Let $V^+ = \{v : w^+(v) > w^-(v)\}$ and $V^- = V \setminus V^+$.

Place any vertex in V^+ in X with probability $(1/2 + p)$. Place any vertex in V^- in Y with probability $(1/2 + p)$.

$$\begin{aligned}\mathbb{E}(w(X,Y)) &= (\frac{1}{4} - p^2)w(D) + (2p^2 + p)w(X,Y) - (p - 2p^2)w(Y,X) \\ &\geq \frac{1}{4}w(D) + (-p^2 + 2p^2\theta(D) + p\theta(D))w(D).\end{aligned}$$

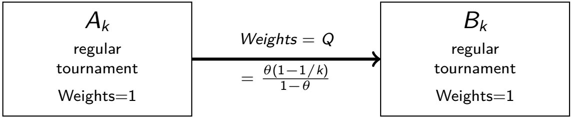
To optimize p we choose $p = \frac{\theta}{2(1-2\theta)}$ and therefore,

$$\mathbb{E}(w(X,Y)) \geq \left(\frac{1}{4} + \frac{\theta^2}{4(1-2\theta)}\right) w(D).$$

Theorem 2 is tight

To show the bound is tight we let D_k be a digraph consisting of two vertex disjoint regular tournament, A_k and B_k, of order k and arc-weights 1.

We then add all arcs from A_k to B_k with weight $Q = \frac{\theta(1-1/k)}{1-\theta}$.



$\theta(D_k) = Qk^2/(k^2 - k + Qk^2) = Q/(1 + Q - 1/k) = \theta$.

For any directed cut $w_{D_k}(X, Y)$,
$w_{D_k}(X, Y) = Qxy + x(k - x)/2 + y(k - y)/2$, where $x = |V(A_k) \cap X|$ and $y = |V(B_k) \cap Y|$ for optimal (X, Y).

$mac(D_k) = \frac{k^2}{4(1-Q)}$ when $0 < Q \leq 1/2$ and $mac(D_k) = Qk^2$ when $Q > 1/2$.

Acyclic digraphs

For general digraphs $mac(D) \geq \frac{m}{4} + \Omega(\sqrt{m})$ (and regular tournaments show that this is bound tight). For (unweighted) acyclic digraphs. Alon et. al. proved the following

Theorem 3 (Alon et al): There exists a constant k_1^s, such that for every integer $m \geq 1$ there exists an acyclic digraph D_m^s with m arcs and $mac(D_m^s) \leq \frac{m}{4} + k_1^s m^{0.8}$.

Theorem 4 (Alon et al): There exists a constant k_2^s, such that $mac(D) \geq \frac{m}{4} + k_2^s m^{0.6}$ for all acyclic digraphs D of size m.

We generalize to multi-digraphs and arc-weighted digraphs.

Theorem 5, [1]: There exists a constant k_1, such that for every integer $m \geq 1$ there exists an acyclic multi-digraph D_m with m arcs and $mac(D_m) \leq \frac{m}{4} + k_1 m^{0.75}$.

Theorem 6, [1]: There exists a constant k_2, such that $mac(D) \geq \frac{w(D)}{4} + k_2 w(D)^{0.6}$ for all acyclic arc-weighted digraphs D with minimum arc weight at least 1.

Acyclic digraphs

Theorem 5 and 6 hold for both multi-digraphs and arc-weighted digraphs with minimum arc weight at least 1.

> Theorem 5: There exists acyclic multi-digraphs: $mac(D_m) \leq \frac{m}{4} + k_1 m^{0.75}$.
> Theorem 6: For all acyclic arc-weighted digraphs with minimum arc weight at least 1. $mac(D) \geq \frac{w(D)}{4} + k_2 w(D)^{0.6}$.

Why do we need it? Otherwise Theorem 6 is not true (choose $w(D)$ sufficiently small such that $k_2 \cdot w(D)^{0.6} > w(D)$).

We first outline the proof of Theorem 5.

Let $V(D) = \{v_1, v_2, \dots, v_n\}$ and add an acyclic tournament on $I_i = (v_i, v_{i+1}, \dots, v_{i+q-1})$ where all arcs go "forward" in the order of I_i and all indices are taken modulo n.

This gives us a regular multi-digraph (where n and q will be decided later).

Theorem 5

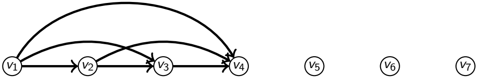
The result we call D_n^*, which is a regular multi-digraph.

We now delete all "backward" arcs and call the result D_n.

As $A(D_n^*)$ can be partitioned into n tournaments on q vertices we note that $mac(UG(D_n^*)) \leq n \cdot mac(K_q) = n \cdot \lfloor \frac{q^2}{4} \rfloor \leq \frac{nq^2}{4}$.

So, $mac(D_n) \leq mac(D_n^*) = \frac{mac(UG(D_n^*))}{2} \leq \frac{nq^2}{8}$.

Theorem 5



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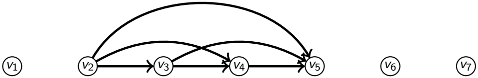

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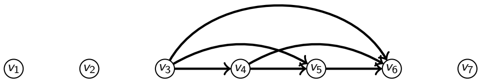

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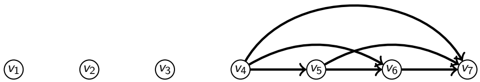

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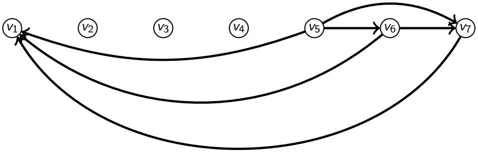

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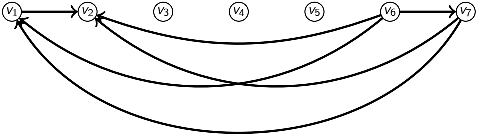

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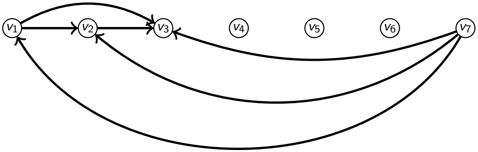

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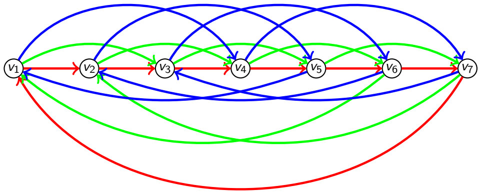

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green ⇒ w=2

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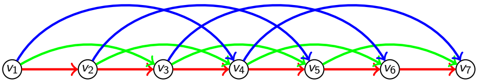

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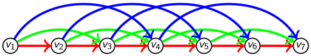
Theorem 5

Example
$n = 7$ and $q = 4$:



$mac(D_n) \leq \frac{nq^2}{8}$.

$$\begin{aligned}|A(D_n)| &= |A(D_n^*)| - 1\cdot(q-1) - 2\cdot(q-2) - \cdots (q-1)\cdot 1 \\ &= n\binom{q}{2} - \sum_{i=1}^{q-1} i(q-i) \\ &= \frac{nq(q-1)}{2} - q\sum_{i=1}^{q-1} i + \sum_{i=1}^{q-1} i^2 \\ &= \ldots = \frac{nq^2}{2} - \frac{nq}{2} - \frac{q^3}{6} + \frac{q}{6}\end{aligned}$$

Letting $q = \lfloor\sqrt{n}\rfloor$ and optimizing we get

$$\begin{aligned}mac(D_n) &\leq \frac{nq^2}{8} = \frac{|A(D_n)|}{4} + \frac{3nq+q^3-q}{24} \\ &\leq \frac{|A(D_n)|}{4} + O(|A(D_n)|^{0.75})\end{aligned}$$

One can then extend this to all values of m....

Theorem 6

Recall Theorem 6, which we shall now give the main ideas for.

Theorem 6, [1]: There exists a constant k_2, such that $mac(D) \geq \frac{w(D)}{4} + k_2 w(D)^{0.6}$ for all arc-weighted acyclic digraphs D with minimum arc weight at least 1.

In order to prove this we need a result on arc-weighted acyclic digraphs with maximum path containing ν vertices.

Let c_ν be the largest number such that $mac(D) \geq c_\nu \times w(D)$ for all arc-weighted acyclic digraphs D with maximum path order at most ν.

Theorem 7, [1]: $c_\nu \geq \frac{1}{4} + \frac{1}{8 \times 3^{2/3} \times \nu^{2/3}}$.

Proving Theorem 7 is the main part in proving Theorem 6.

We can show that $c_2 = 1$, $c_3 = c_4 = \frac{1}{2}$, $c_5 = c_6 = \frac{2}{5}$, $c_7 = \frac{3}{8}$, $c_8 = \frac{4}{11}$, $c_9 = \frac{13}{37}$, $c_{10} = \frac{9}{26}$ and $c_{11} = \frac{31}{92}$.

Theorem 7

Let D be an arc-weighted acyclic digraph with maximum path order ν.

Theorem 7: $c_\nu \geq \frac{1}{4} + \frac{1}{8 \times 3^{2/3} \times \nu^{2/3}}$.

There exists independent sets $S_1, S_2, \ldots, S_\nu$ such that all arcs in D are (S_i, S_j)-arcs with $i < j$.

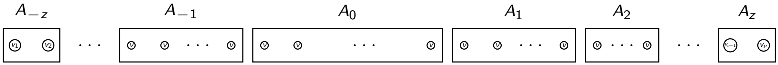
We contract each S_i into a vertex v_i, which gives us an acyclic digraph D' with $V(D') = \{v_1, v_2, \ldots, v_\nu\}$.

Every directed cut of the reduced digraph corresponding to a directed cut of the original digraph.

Lemma

For every digraph D with $2n$ vertices there is a random directed cut (X, Y) such that every arc in D is included in (X, Y) with probability $\frac{n}{4n-2}$.

Theorem 7: Intuitions



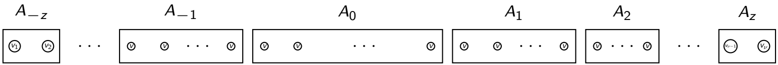
Let $|A_0| = 2k$ where $k = O(\nu^{2/3})$. Then, by the lemma from the last page there is a random partition (X_0, Y_0) of A_0 such that every arc is included in (X_0, Y_0) with probability at least $\frac{k}{4k-2}$.

Goal: generalize it to a random directed cut (X, Y) of the whole digrph D such that every arc is in (X, Y) with probability at least $\frac{k}{4k-2}$.

(1) vertices in A_i should be in Y with a increasing probability as i increase when $i > 0$.

(2) for $i > 0$ vertices in $|A_i| > |A_{i+1}|$ and $|A_{-i}| > |A_{-(i+1)}|$.

Theorem 7

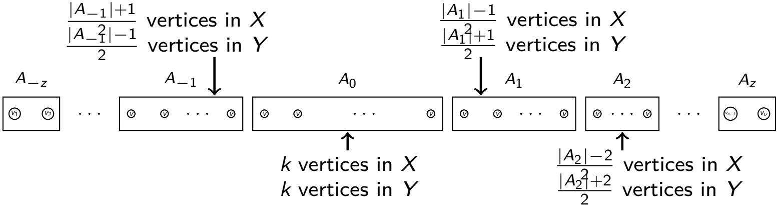


We then (for some k) partition the vertices into sets $A_{-z}, A_{-z+1}, \ldots, A_{-1}, A_0, A_1, \ldots, A_z$ ($z = \lfloor \sqrt{k/2} \rfloor$), such that $|A_i| = 2k - 2i^2$ for all $i \in \{-z, -z+1, \ldots, z\}$.

We randomly place $\frac{|A_i|-i}{2}$ vertices from A_i in X and $\frac{|A_i|+i}{2}$ vertices from A_i in Y when $i \geq 0$.

We randomly place $\frac{|A_i|+i}{2}$ vertices from A_i in X and $\frac{|A_i|-i}{2}$ vertices from A_i in Y when $i < 0$.

Theorem 7



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Theorem 7

If a is an arc within A_i, then

To show: $P(a \in (X, Y)) \geq \frac{k}{4k-2}$

Claim A. $P(a \in (X, Y)) \geq \frac{k}{4k-2}$ for every $a \in D'$.

Proof.

□

Analogously, one can then show that it also holds for arcs between sets.

Theorem 7

One can also show that $\nu \geq k^{3/2}$. So,

$$\begin{aligned} mac(D) &\geq \frac{k}{4k-2} \times w(D) \\ &= \left(\frac{1}{4} + \frac{1}{8k-4}\right) w(D) \\ &\geq \left(\frac{1}{4} + \frac{1}{8\nu^{2/3}-4}\right) w(D) \end{aligned}$$

In Theorem 7 we show $mac(D) \geq \left(\frac{1}{4} + \frac{1}{8\times 3^{2/3}\times \nu^{2/3}}\right) w(D)$, which is because we need the bound to hold for all ν, not just the ones we consider above.

Theorem 6

Theorem 6: There exists a constant k_2, such that $mac(D) \geq \frac{w(D)}{4} + k_2 w(D)^{0.6}$ for all arc-weighted acyclic digraphs D with minimum arc weight at least 1.

> Theorem 7:
>
> $c_\nu \geq \frac{1}{4} + \frac{1}{8\times 3^{2/3}\times \nu^{2/3}}$.

Proof: Let D be a arc-weighted acyclic digraphs D.

Let $P = p_1 p_2 p_3 \dots p_n$ be a longest path in D.

We consider the cases when $w(P) \leq w(D)^{0.6}$ and $w(P) \geq w(D)^{0.6}$ seperately.

Case 1: $w(P) \leq w(D)^{0.6}$.

As all weights are at least one, we have $|A(P)| \leq w(P) \leq w(D)^{0.6}$. So Theorem 7 implies,

$$\begin{aligned} mac(D) &\geq \left(\frac{1}{4} + \frac{1}{8\times 3^{2/3}\times |A(P)|^{2/3}}\right) w(D) \\ &\geq \frac{w(D)}{4} + \frac{w(D)}{8\times 3^{2/3}\times w(D)^{0.4}} \\ &\geq \frac{w(D)}{4} + k_2 \cdot w(D)^{0.6} \end{aligned}$$

Theorem 6, Case 2 proof

Case 2: $w(P) \geq w(D)^{0.6}$.

Small cycles

Let *circ*(D) denote the *circumference* of a digraph D (i.e. the length of a longest cycle in D). Assume all weights are at least 1.

Theorem 8, [1]: Assume that there exist constants $k > 0$ and $0 < \alpha < 1$ such that $\mathrm{mac}(H) \geq \frac{w(H)}{4} + kw(H)^{\alpha}$ for all acyclic digraphs H. If D is an arbitrary arc-weighted digraph then,

$$\mathrm{mac}(D) \geq \frac{w(D)}{4} + \frac{k}{(4k+1) \cdot circ(D) + 1} \times w(D)^{\alpha}$$

The proof of Theorem 8 uses the following theorem.

Theorem 9 (Bondy, 1976): For all strong digraphs D we have $\chi(D) \leq circ(D)$.

So, any result holding for acyclic digraphs also holds for digraphs where the circumference is bounded by a constant.

Open problem

Theorem 5, [1]: There exists a constant k_1, such that for every integer $m \geq 1$ there exists an acyclic multi-digraph D_m with m arcs and $mac(D_m) \leq \frac{m}{4} + k_1 m^{0.75}$.

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Open Problem: Close the gap between 0.6 and 0.75 for arc-weighted acyclic digraphs D.

Final open problem

For simple digraphs the following holds.

Theorem 3 (Alon et al): There exists a constant k_1^s, such that for every integer $m \geq 1$ there exists an acyclic digraph D_m^s with m arcs and $mac(D_m^s) \leq \frac{m}{4} + k_1^s m^{0.8}$.

Theorem 4 (Alon et al): There exists a constant k_2^s, such that $mac(D) \geq \frac{m}{4} + k_2^s m^{0.6}$ for all acyclic digraphs D of size m.

Open Problem: Close the gap between 0.6 and 0.8 for simple acyclic digraphs D.

End of the talk

This completes the talk, which was based on the paper

[1] Jiangdong Ai, Stefanie Gerke, Gregory Gutin, Anders Yeo and Yacong Zhou. *Bounds on Maximum Weight Directed Cut.* SIAM Journal on Discrete Math., Vol. 38, Iss. 3 (2024).

The End

Thank you for the invitation to come here.

And thank you to the organizers for doing a great job.

Any questions?