Lower Bounds for Maximum Weight Bisections

of Graphs with Bounded Degrees

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Cuts, Bisections and their weights

Definition

A cut of a weighted graph $G = (V, E, w)$ is a partition of the vertex set V into two disjoint subsets X and Y. The weight of the cut, denoted by $w(X, Y)$, refers to the sum of the weights over all edges between X and Y.

Definition

A bisection of G is a cut (X, Y) where the cardinality of X and Y differs by at most one.

Theorem ([P. Erdős, 1965])

Every graph $G = (V(G), E(G))$ has a cut with at least $\frac{|E(G)|}{2}$ edges.

Lemma

Let $G = (V, E, w)$ be a weighted complete graph with n vertices. Then, G has a bisection with weight at least $\frac{n}{2(n-1)} w(G)$ when n is even and $\frac{n+1}{2n} w(G)$ when n is odd.

Theorem ([L. Anderson, D. Grant and N. Linial, 1983, J. Lehel and Zs. Tuza, 1982, S. Locke, 1982])

Let $G = (V(G), E(G), w)$ be a weighted graph and $\chi = \chi(G)$. Then, G admits a cut of weight at least $\frac{\chi+1}{2\chi}w(G)$ when χ is odd and $\frac{\chi}{2(\chi-1)}w(G)$ when χ is even.

By Brook's theorem[1], we have the following holds.

Corollary

Let $G = (V(G), E(G), w)$ be a weighted graph. If $\Delta(G) \leq k$, then there exists a cut of weight at least $\frac{k+1}{2k}w(G)$ if k is odd and $\frac{k+2}{2(k+1)}w(G)$ if k is even.

[1] $\chi(G) \leq \Delta(G) + 1$

Our conjecture and past results

Conjecture ([S. Gerke, G. Gutin, A. Yeo and Y. Zhou, 2024+])

Let $G = (V(G), E(G), w)$ be a weighted graph. If $\Delta(G) \leq k$, then there exists a bisection of weight at least $\frac{k+1}{2k}w(G)$ if k is odd and $\frac{k+2}{2(k+1)}w(G)$ if k is even.

Theorem ([B. Bollobás and A. Scott, 2004])

Let $G = (V(G), E(G))$ be a graph and k be an odd integer. If G is k-regular, then there exists a bisection of size at least $\frac{k+1}{2k}|E(G)|$.

Theorem ([C. Lee, P. Loh and B. Sudakov, 2013])

Let $G = (V(G), E(G))$ be a graph. If $\Delta(G) \leq k$, then there exists a bisection of size at least $\frac{k+1}{2k}|E(G)| - \frac{k(k+1)}{4}$ if k is odd and $\frac{k+2}{2(k+1)}|E(G)| - \frac{k(k+2)}{4}$ if k is even.

The Key lemma

We say that $B \in \mathcal{B}_b(G)$ or B is a (partially) induced balanced bipartite subgraph of G, if B is the union of vertex-disjoint bipartite subgraphs B_i's (not necessarily connected) of G with partite sets (X_i, Y_i) where $G[X_i]$ and $G[Y_i]$ have no edges and $|X_i| = |Y_i|$.

Lemma

Let $G = (V, E, w)$ be a weighted graph and $B \in \mathcal{B}_b(G)$. Then, G has a bisection with weight at least $\frac{w(G)+w(B)}{2}$.

Since any matching M of G is clearly in $\mathcal{B}_b(G)$, we have the following corollary.

Corollary

Let $G = (V, E, w)$ be a weighted graph and M its maximum weight matching. Then, G has a bisection with weight at least $\frac{w(G)+w(M)}{2}$.

Theorem

Every weighted graph $G = (V, E, w)$ has a bisection with weight at least $\frac{\chi'(G)+1}{2\chi'(G)} w(G)$.

Bounds for Maximum weight bisection

of Graphs with even maximum degrees

The following bound for chromatic index is known as Vizing's Theorem.

Theorem ([V. G. Vizing, 1964])

For any simple graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

Corollary

Let k be a positive integer. Any weighted graph G with $\Delta(G) \leq k$ has a bisection with weight at least

$$\frac{k+2}{2(k+1)} w(G).$$

In particular, the conjecture holds when k is even.

Theorem ([S. Gerke, G. Gutin, A. Yeo and Y. Zhou, 2024+])

Every weighted subcubic graph G has a bisection with weight at least $\frac{2}{3}w(G)$.

We may assume that G has at most one vertex with degree not equal to 3 as we may add edges of weight 0 between any two vertices with degree less than 3. We consider three cases.

Theorem ([D. Mattiolo and G. Mazzuoccolo, 2021])

Every cubic multigraph has a 3-bisection.

Lemma

Every weighted cubic multigraph G has a bisection (X, Y) such that the following holds.

(I) $G[X] \cup G[Y]$ *is a forest;*

(II) $|(X, Y)|$ *attains the maximum value among all bisection that satisfy (I);*

(III) $\Delta(G[X]) \leq 1$ *and* $|E(G[Y])| \leq |Y|/2$.

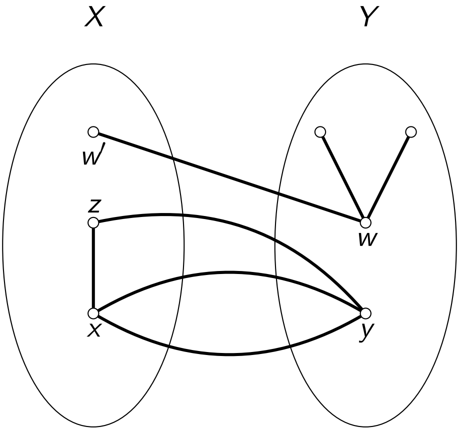
Sketch of the Proof

One vertex with degree two, say $d_G(z) = 2$

Add an vertex x and edge xz with weight 0.

Sketch of the Proof

One vertex with degree one, say $d_G(z) = 1$



Theorem
([S. Gerke, G. Gutin, A. Yeo and Y. Zhou, 2024+])

Every triangle-free subcubic graph G has a bisection with weight at least $\frac{613}{855} \cdot w(G)$ unless $G \cong K_{1,3}$ (where $\frac{613}{855} \approx 0.716959$).

Our conjecture

Conjecture
([S. Gerke, G. Gutin, A. Yeo and Y. Zhou, 2024+])

Every triangle-free subcubic graph G has a bisection with weight at least $\frac{11}{15} \cdot w(G)$ unless $G \cong K_{1,3}$ (where $\frac{11}{15} \approx 0.733333$).

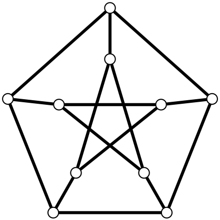


Figure: The Petersen graph

Thank you for your attention!

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